A CONJECTURE OF WATKINS FOR QUADRATIC TWISTS

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Abstract. Watkins conjectured that for an elliptic curve E over $\mathbb{Q}$ of Mordell-Weil rank r, the modular degree of E is divisible by 2^r. If E has non-trivial rational 2-torsion, we prove the conjecture for all the quadratic twists of E by squarefree integers with sufficiently many prime factors.

1. Ranks and modular degree

For an elliptic curve E over $\mathbb{Q}$ of conductor N, the modularity theorem [25, 23, 4] gives a non-constant morphism $\phi_E : X_0(N) \to E$ defined over $\mathbb{Q}$ where $X_0(N)$ is the modular curve associated to the congruence subgroup $\Gamma_0(N) \subseteq SL_2(\mathbb{Z})$. We assume that ϕ_E has minimal degree and that it maps the cusp $i\infty$ to the neutral point of E. These requirements uniquely determine ϕ_E up to sign. The *modular degree* of E is $m_E = \deg \phi_E$ and it has profound arithmetic relevance; for instance, polynomial bounds for its size in terms of N are essentially equivalent to the *abc* conjecture [11, 17].

The 2-adic valuation is denoted by v_2. Motivated by numerical data, Watkins [24] conjectured that $v_2(m_E)$ for an elliptic curve E is closely related to the Mordell-Weil rank of E over $\mathbb{Q}$.

Conjecture 1.1 (Watkins)**.** *For every elliptic curve E over $\mathbb{Q}$ we have* $\operatorname{rank} E(\mathbb{Q}) \le v_2(m_E)$.

Dummigan [8] showed that part of the conjecture would follow from strong $R = \mathbb{T}$ conjectures. Also, large part of Watkins' conjecture is proved for elliptic curves of odd modular degree [5, 26, 12, 13], although it is not known whether there exist infinitely many elliptic curves of this kind [21].

The goal of this note is to prove Watkins' conjecture unconditionally in several new cases. Let us introduce some notation. For an elliptic curve E and a fundamental (quadratic) discriminant D, the quadratic twist of E by D is denoted by $E^{(D)}$. The Manin constant of E is denoted by c_E (cf. Section 2.3). The number of distinct prime factors of an integer n is $\omega(n)$.

Theorem 1.2. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N with non-trivial rational 2-torsion. Assume that E has minimal conductor among its quadratic twists. If D is a fundamental discriminant with $\omega(D) \ge 6 + 5\omega(N) - v_2(m_E/c_E^2)$, then Watkins' conjecture holds for $E^{(D)}$.*

The quantity $6 + 5\omega(N) - v_2(m_E/c_E^2)$ is effectively computable and it can be read from existing tables of elliptic curves when N is not too large, see for instance [14].

For a positive integer A, it is a standard result of analytic number theory that the number of positive integers n up to x having $\omega(n) \le A$ is $O(x(\log\log x)^{A-1}/\log x)$. We deduce:

Corollary 1.3. *Let E be an elliptic curve over $\mathbb{Q}$ with non-trivial rational 2-torsion. There is an effective constant $\kappa(E)$ depending only on E such that the number of fundamental discriminants D with $|D| \le x$ such that Watkins' conjecture fails for $E^{(D)}$ is bounded by* $O\left(x(\log\log x)^{\kappa(E)}/\log x\right)$.

Let us remark that in the cases where we prove Watkins' conjecture our argument actually shows that $v_2(m_{E^{(D)}})$ bounds the 2-Selmer rank, which is a stronger version of Watkins' conjecture.

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2. Preliminaries

2.1. **Faltings height.** Let E be an elliptic curve over $\mathbb{Q}$. We denote by ω_E a global Neron differential for E; it is unique up to sign. The Faltings height of E (over $\mathbb{Q}$) is defined as certain Arakelov degree [10], which in our case takes the simpler form [19]

$$h(E) = -\frac{1}{2}\log\left(\frac{i}{2}\int_{E(\mathbb{C})}\omega_E\wedge\overline{\omega_E}\right). \tag{2.1}$$

Ramanujan's cusp form is $\Delta(z) = q\prod_{n=1}^{\infty}(1-q^n)^{24}$ where $q=\exp(2\pi iz)$, defined on the upper half plane $\mathfrak{h} = \{z\in\mathbb{C} : \Im(z)>0\}$. The modular j-function is normalized as $j(z) = q^{-1}+744+...$

The global minimal discriminant of E is denoted by Δ_E. If $\tau_E\in\mathfrak{h}$ satisfies that $j(\tau_E)$ is the j-invariant of E, then the Faltings height admits the expression [22, 19]

$$h(E) = \frac{1}{12}\left(\log|\Delta_E| - \log\left|\Delta(\tau_E)\Im(\tau_E)^6\right|\right) - \log(2\pi). \tag{2.2}$$

Given elliptic curves E_1, E_2 over $\mathbb{Q}$, let us define $\delta(E_1,E_2) = \exp(2h(E_1)-2h(E_2))$.

Lemma 2.1 (Variation of $h(E)$ under quadratic twist)**.** *Let E_1 be an elliptic curve over $\mathbb{Q}$ and let E_2 be a quadratic twist of E_1. Then $\delta(E_1,E_2)$ is a rational number and it satisfies $|v_2(\delta(E_1,E_2))|\le 3$.*

Proof. We use (2.2) for both E_1 and E_2. The elliptic curves are isomorphic over $\mathbb{C}$, so we can take $\tau_{E_1}=\tau_{E_2}$ which gives $\delta(E_1,E_2) = |\Delta_{E_1}/\Delta_{E_2}|^{1/6}$. The result follows from explicit formulas for the variation of the minimal discriminant under quadratic twists, cf. Proposition 2.4 in [18]. □

2.2. **Petersson norm.** For a positive integer N, let $S_2(N)$ be the space of weight 2 cuspidal holomorphic modular forms for the congruence subgroup $\Gamma_0(N)$ acting on $\mathfrak{h}$. Given $f\in S_2(N)$, its Fourier expansion is $f(z) = a_1(f)q + a_2(f)q^2+...$ where $q=\exp(2\pi iz)$ and the numbers $a_n(f)$ are the Fourier coefficients of f. The Petersson norm of f relative to $\Gamma_0(N)$ is defined by

$$\|f\|_N = \left(\int_{\Gamma_0(N)\backslash\mathfrak{h}}|f(z)|^2dx\wedge dy\right)^{1/2}, \quad z=x+iy\in\mathfrak{h}.$$

The norm depends on the choice of N in the following sense: If $N|M$ and $f\in S_2(N)$, then we certainly have $f\in S_2(M)$, and $\|f\|_M^2 = [\Gamma_0(N):\Gamma_0(M)]\cdot\|f\|_N^2$.

We need some additional notation. For an elliptic curve E over $\mathbb{Q}$ of conductor N we denote by $f_E\in S_2(N)$ the Hecke newform attached to E by the modularity theorem, normalized by $a_1(f_E)=1$. The modular form f_E is characterized by the following property: If p is a prime of good reduction for E and we define $a_p(E) = p+1-\#E(\mathbb{F}_p)$, then $a_p(f_E) = a_p(E)$. For a fundamental discriminant D, let $\mathscr{P}(D,N)$ be the set of primes p with $p|D$ and $p\nmid 2N$.

Lemma 2.2 (Variation of the Petersson norm under quadratic twist)**.** *Let E be an elliptic curve over $\mathbb{Q}$ and let D be a fundamental discriminant. Let N and $N^{(D)}$ be the conductors of E and $E^{(D)}$ respectively, and assume that $N|N^{(D)}$. Then $\|f_{E^{(D)}}\|^2_{N^{(D)}}/\|f_E\|_N^2\in\mathbb{Q}^\times$ and we have*

$$v_2(\|f_{E^{(D)}}\|^2_{N^{(D)}}/\|f_E\|^2_N)+1 \ge \sum_{p\in\mathscr{P}(D,N)} v_2\left((p-1)(p+1-a_p(E))(p+1+a_p(E))\right).$$

Proof. The quadratic Dirichlet character attached to D has conductor $|D|$. The result follows from the precise formula given in Theorem 1 of [7] when one only keeps the contribution of $p=2$ and the primes $p\in\mathscr{P}(D,N)$ —the product of the latter primes is denoted by D_1 in *loc. cit.* □

We remark that the terms $(p-1)(p+1-a_p(E))(p+1+a_p(E))$ have a clear conceptual origin; they come from Euler factors of the imprimitive symmetric square L-function $L(\mathrm{Sym}^2 f_E, s)$ that are removed by twisting, and $L(\mathrm{Sym}^2 f_E, 2)$ is (up to a mild factor) equal to $\|f_E\|_N^2$. See [27, 7, 24].

2.3. **Manin constant.** Given an elliptic curve E over $\mathbb{Q}$ of conductor N, we have that $\phi_E^*\omega_E$ is a regular differential on $X_0(N) = \Gamma_0(N)\backslash\mathfrak{h} \cup \{\text{cusps}\}$. More precisely

$$\phi_E^*\omega_E = 2\pi i c_E f_E(z)dz \tag{2.3}$$

where c_E is a rational number uniquely defined up to sign. We assume that the signs of ϕ_E and ω_E are chosen such that $c_E > 0$. It follows from (2.1) and (2.3) that (cf. [22, 19])

$$m_E = 4\pi^2 c_E^2 \|f_E\|_N^2 \exp(2h(E)). \tag{2.4}$$

The quantity c_E is called the Manin constant, and a fundamental fact is

Lemma 2.3 (cf. [9])**.** *The Manin constant c_E is an integer.*

We recall that Manin [15] conjectured that if E is a strong Weil curve in the sense that m_E is minimal within the isogeny class of E, then $c_E = 1$. See [16, 3, 2, 6] and the references therein.

3. Consequences for Watkins' conjecture

Lemma 3.1. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and suppose that E has minimal conductor among its quadratic twists. Let D be a fundamental discriminant. Then*

$$v_2(m_{E^{(D)}}) \geq v_2(m_E/c_E^2) - 4 + \sum_{p\in\mathscr{P}(D,N)} v_2\left((p-1)(p+1-a_p(E))(p+1+a_p(E))\right).$$

Proof. Applying (2.4) to E and $E^{(D)}$ we find

$$\frac{m_{E^{(D)}}}{m_E} = \frac{c_{E^{(D)}}^2}{c_E^2} \cdot \frac{\|f_{E^{(D)}}\|_{N^{(D)}}^2}{\|f_E\|_N^2} \cdot \delta(E^{(D)}, E).$$

The result follows from lemmas 2.1, 2.2, and 2.3. □

Proposition 3.2. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N with non-trivial rational 2-torsion and suppose that E has minimal conductor among its quadratic twists. Let D be a fundamental discriminant. We have $v_2(m_{E^{(D)}}) \geq 3\omega(D) + v_2(m_E/c_E^2) - (7 + 3\omega(N))$.*

Proof. As $E(\mathbb{Q})[2]$ is non-trivial and it maps injectively into $E(\mathbb{F}_p)$ for every prime $p \nmid 2N$, we have $p + 1 \equiv a_p(E) \bmod 2$ for these primes. We get $v_2(m_{E^{(D)}}) \geq v_2(m_E/c_E^2) - 4 + 3 \cdot \#\mathscr{P}(D,N)$ from Lemma 3.1, and the result follows from $\#\mathscr{P}(D,N) \geq \omega(D) - \omega(2N) \geq \omega(D) - \omega(N) - 1$. □

The following upper bound for the Mordell-Weil rank is standard and it comes from a bound for a 2-isogeny Selmer rank (cf. Section X.4 in [20]; see also [1]).

Lemma 3.3. *Let E be an elliptic curve over $\mathbb{Q}$ of conductor N with non-trivial rational 2-torsion. Then* $\operatorname{rank} E(\mathbb{Q}) \leq 2\omega(N) - 1$.

Proof of Theorem 1.2. Since $E^{(D)}[2] \simeq E[2]$ as Galois modules and E has non-trivial rational 2-torsion, we can use Lemma 3.3 for $E^{(D)}$, which gives

$$\operatorname{rank} E^{(D)}(\mathbb{Q}) \leq 2\omega(N^{(D)}) - 1 \leq 2(\omega(D) + \omega(N)) - 1.$$

If Watkins' conjecture fails for $E^{(D)}$, then Proposition 3.2 would give

$$2(\omega(D) + \omega(N)) - 1 \geq v_2(m_{E^{(D)}}) + 1 \geq 3\omega(D) + v_2(m_E/c_E^2) - 6 - 3\omega(N).$$

This is not possible when $\omega(D) \geq 6 + 5\omega(N) - v_2(m_E/c_E^2)$. □

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